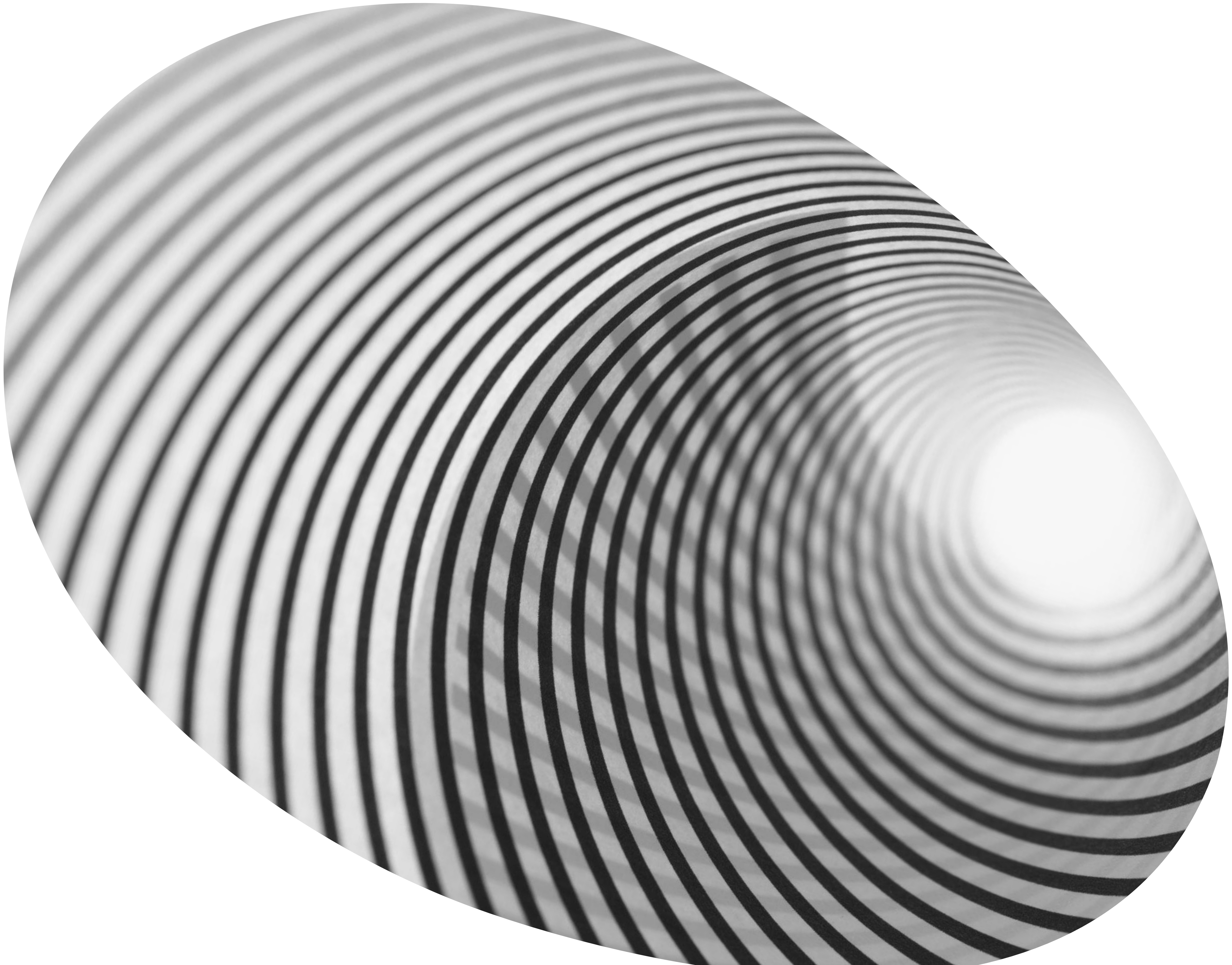


Day 01 – The many Facets of Quantum Topology

NQIS Summer School

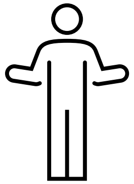
July 18th & 22nd

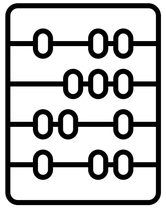
1:30—3:30

Eugene Dumitrescu (ORNL)

Assisted by Yan Wang (ORNL)

Topological Quantum Computation Algorithms





Q: What does this mean?

- The underlying topology of quantum computing algorithms?
- Algorithms to topologically error-corrected quantum computers?
- Quantum algorithms for computing topological quantities?
- Classical algorithms to compute the topology of hardware taking part in quantum computations?
  - Develop a topological qubit? Create analog/synthetic topological matter? Program in a topologically error corrected code-space?

A: All are valid interpretations.

- Our goal is to differentiate the facets of topological quantum computing algorithms (TQCA) so that you can go beyond this and say *what you mean*.
- To do so, we must first define *quantum topology*.

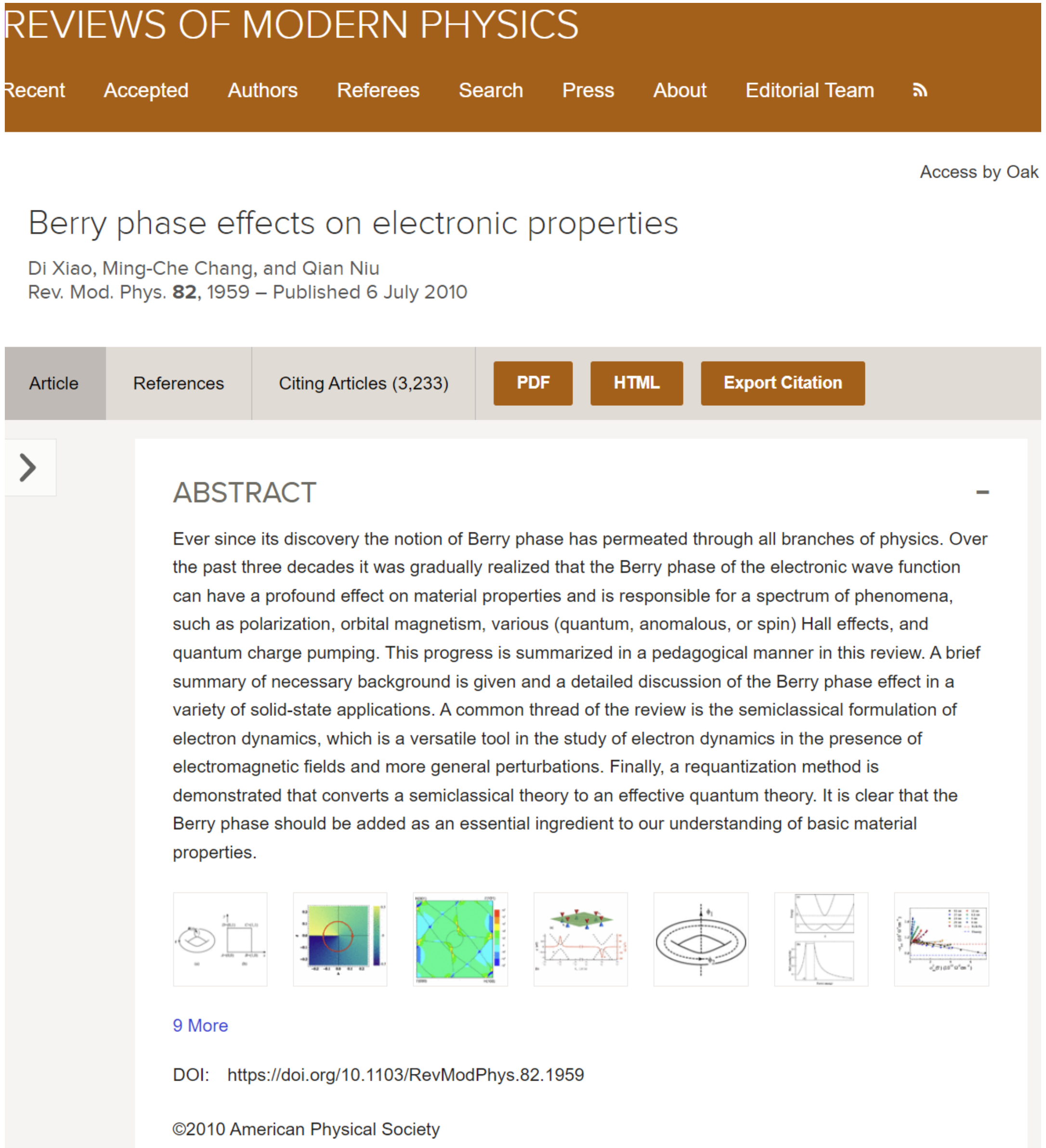
Day I:(Classical and) Quantum Topology

The Nobel Prize in Physics 2016
David J. Thouless, F. Duncan M. Haldane and J. Michael Kosterlitz
“for theoretical discoveries of topological phase transitions and topological phases of matter”

The Nobel Prize in Physics 1998
Robert B. Laughlin, Horst L. Störmer and Daniel C. Tsui
“for their discovery of a new form of quantum fluid with fractionally charged excitations”

The Nobel Prize in Physics 1985
Klaus von Klitzing
“for the discovery of the quantized Hall effect”

Definitions
Examples
Mathematical Applications



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Access by Oak

Berry phase effects on electronic properties

Di Xiao, Ming-Che Chang, and Qian Niu
Rev. Mod. Phys. **82**, 1959 – Published 6 July 2010

Article | References | Citing Articles (3,233) | PDF | HTML | Export Citation

ABSTRACT

Ever since its discovery the notion of Berry phase has permeated through all branches of physics. Over the past three decades it was gradually realized that the Berry phase of the electronic wave function can have a profound effect on material properties and is responsible for a spectrum of phenomena, such as polarization, orbital magnetism, various (quantum, anomalous, or spin) Hall effects, and quantum charge pumping. This progress is summarized in a pedagogical manner in this review. A brief summary of necessary background is given and a detailed discussion of the Berry phase effect in a variety of solid-state applications. A common thread of the review is the semiclassical formulation of electron dynamics, which is a versatile tool in the study of electron dynamics in the presence of electromagnetic fields and more general perturbations. Finally, a requantization method is demonstrated that converts a semiclassical theory to an effective quantum theory. It is clear that the Berry phase should be added as an essential ingredient to our understanding of basic material properties.

9 More

DOI: https://doi.org/10.1103/RevModPhys.82.1959

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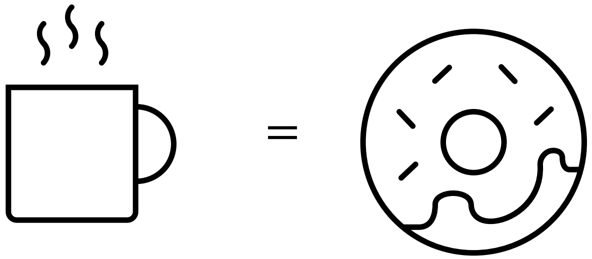
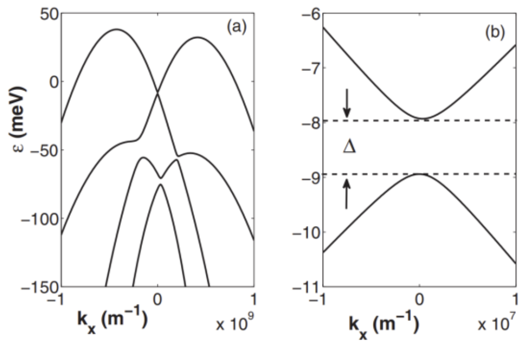
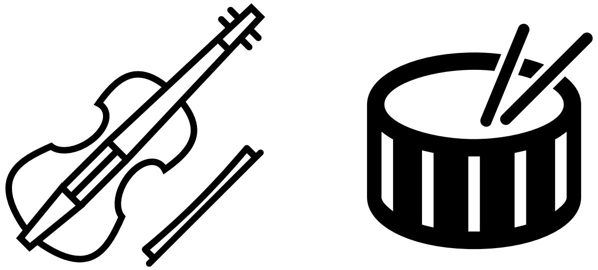
Day 1 – Quantum Topology – Outline

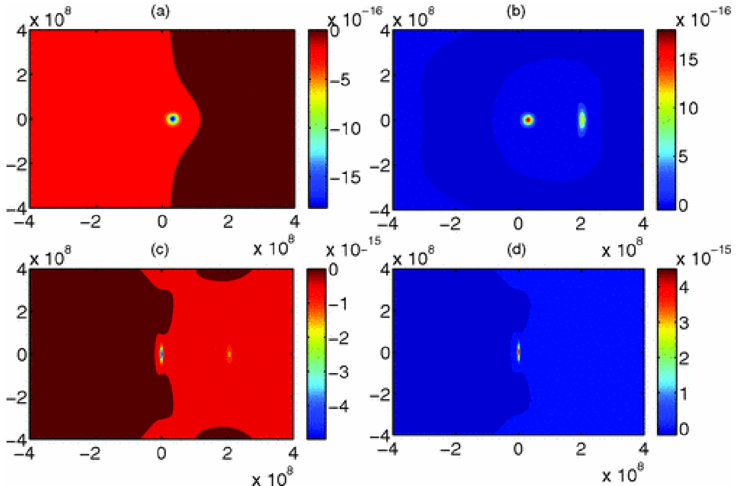
```
def quantum_topology():
```

- Classical Topology:
  - Manifolds
  - Curvature
  - Invariants
  - Hands on Examples (hopefully)
- Classical-quantum Topology
  - Topological defects and excitations in (topologically trivial) quantum systems
- Topologically non-Trivial Quantum Matter
  - Topological Phases, Protected Edge-modes, and the Holographic Duality
  - Examples, Tools & Invariants









Topology

$$\iint_M K\, dA = 2\pi\chi(M)$$


=

=

=


+ all smooth deformations, which preserve topology


=

=

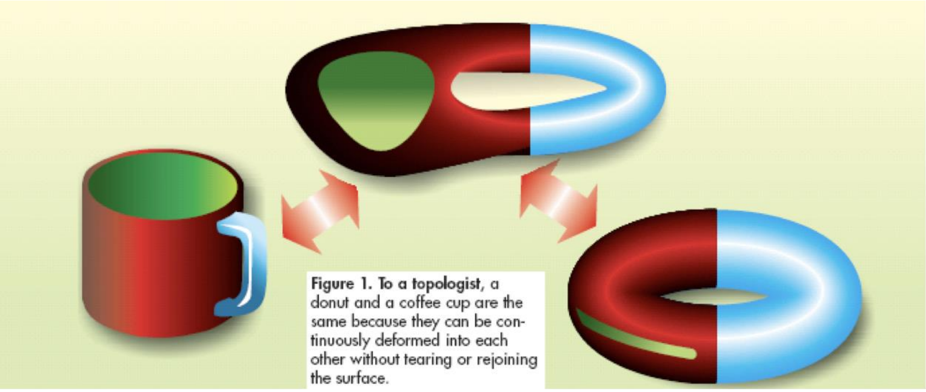
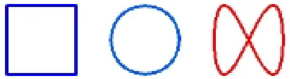
=


Figure 1. To a topologist, a donut and a coffee cup are the same because they can be continuously deformed into each other without tearing or rejoining the surface.

Genus is a **global** property

What is Topology?

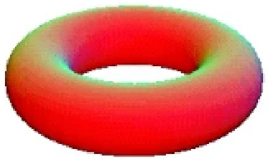
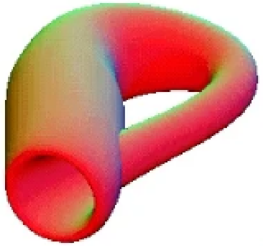
Topology studies properties of spaces that are invariant under any continuous deformation. It is sometimes called "rubber-sheet geometry" because the objects can be stretched and contracted like rubber, but cannot be broken. For example, a square can be deformed into a circle without breaking it, but a figure 8 cannot. Hence a square is topologically equivalent to a circle, but different from a figure 8.



Here are some examples of typical questions in topology: How many holes are there in an object? How can you define the holes in a torus or sphere? What is the boundary of an object? Is a space connected? Does every continuous function from the space to itself have a fixed point?

Topology is a relatively new branch of mathematics; most of the research in topology has been done since 1900. The following are some of the subfields of topology.

1. ***General Topology or Point Set Topology.*** General topology normally considers local properties of spaces, and is closely related to analysis. It generalizes the concept of continuity to define topological spaces, in which limits of sequences can be considered. Sometimes distances can be defined in these spaces, in which case they are called metric spaces; sometimes no concept of distance makes sense.
2. ***Combinatorial Topology.*** Combinatorial topology considers the global properties of spaces, built up from a network of vertices, edges, and faces. This is the oldest branch of topology, and dates back to Euler. It has been shown that topologically equivalent spaces have the same numerical invariant, which we now call the Euler characteristic. This is the number (V - E + F), where V, E, and F are the number of vertices, edges, and faces of an object. For example, a tetrahedron and a cube are topologically equivalent to a sphere, and any "triangulation" of a sphere will have an Euler characteristic of 2.
3. ***Algebraic Topology.*** Algebraic topology also considers the global properties of spaces, and uses algebraic objects such as groups and rings to answer topological questions. Algebraic topology converts a topological problem into an algebraic problem that is hopefully easier to solve. For example, a group called a homology group can be associated to each space, and the torus and the Klein bottle can be distinguished from each other because they have different homology groups.

	
Torus	Klein Bottle

Algebraic topology sometimes uses the combinatorial structure of a space to calculate the various groups associated to that space.

4. ***Differential Topology.*** Differential topology considers spaces with some kind of smoothness associated to each point. In this case, the square and the circle would not be smoothly (or differentiably) equivalent to each other. Differential topology is useful for studying properties of vector fields, such as a magnetic or electric fields.

Topology is used in many branches of mathematics, such as differentiable equations, dynamical systems, knot theory, and Riemann surfaces in complex analysis. It is also used in string theory in physics, and for describing the space-time structure of universe.

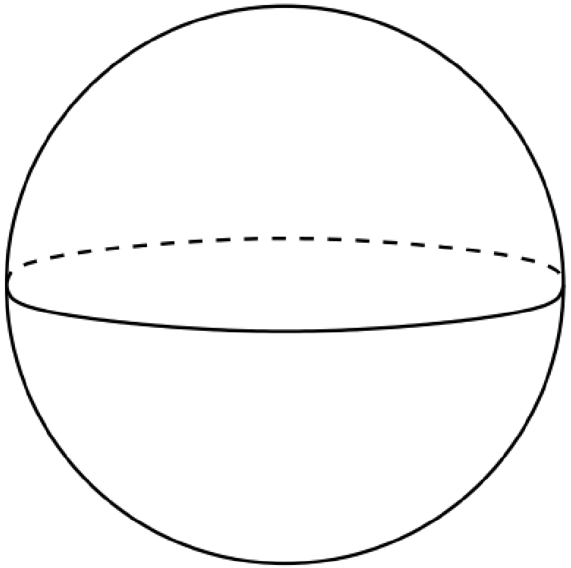
https://uwaterloo.ca/pure-mathematics/about-pure-math/what-is-pure-math/what-is-topology

Birth of the topology:
Seven Bridges of Königsberg solution by Euler

Wednesday July 17
Error correction Eddie Schoute (LANL)
Topological quantum computing from theory/algorithms perspective Eugene Dumitrescu
Thursday July 18
Introduction to QSC landscape Vivien Zapf (LANL) 45 min
Topological spin liquids I Gabor Halasz (ORNL) 45 min
Topological spin liquids II Alan Tennant (ORNL) 45 min
Topological superconductors II Rob Moore (ORNL) 45 min

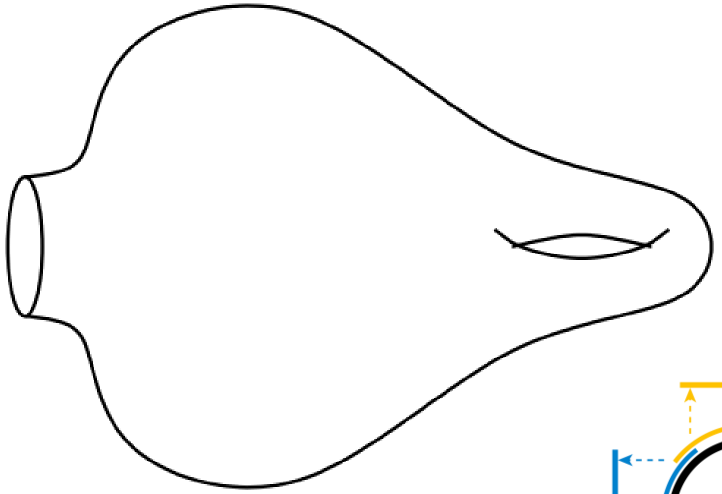
Object studied in physics

“Topology (from the Greek words τόπος, 'place, location', and λόγος, 'study') is the part of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.” -Dr. Wikipedia



M. Vonk (2005) arXiv:hep-th/0504147

(a)



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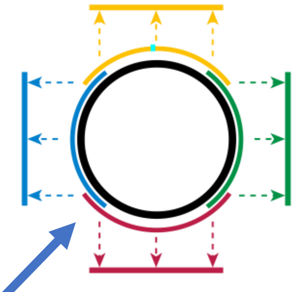
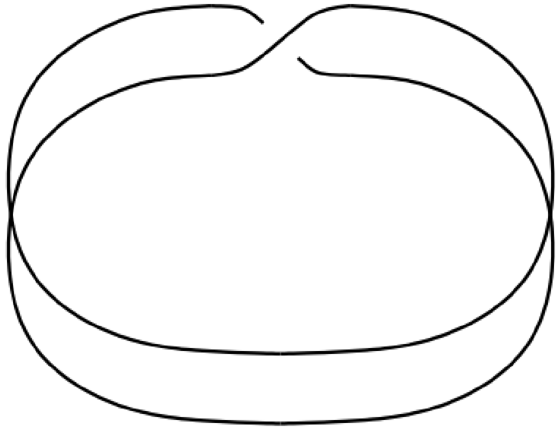
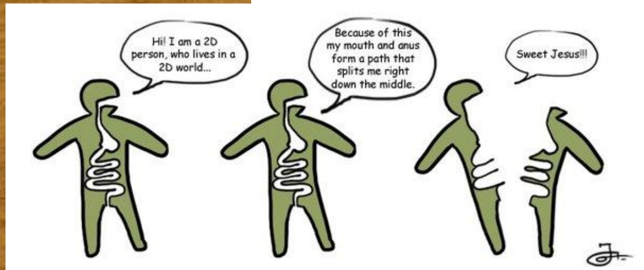




(c)

Figure 3: (a) The sphere is an orientable manifold without handles or boundaries. (b) An orientable manifold with one boundary and one handle. (c) The Möbius strip: an unorientable manifold with one boundary and no handles.


Hi! I am a 2D person, who lives in a 2D world...
Because of this my mouth and anus form a path that splits me right down the middle.
Sweet Jesus!!!

Braid Group: a topological group

σ_1 σ_1^{-1}

is different from

On the other hand, two such connections which can be made to look the same by "pulling the strands" are considered *the same* braid:

smooth deformations

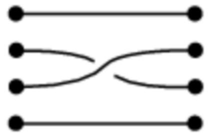
is the same as

All strands are required to move from left to right; knots like the following are *not* considered braids:

is not a braid

Any two braids can be *composed* by drawing the first next to the second, identifying the four items in the middle, and connecting corresponding strands:

composed with



yields

Another example:

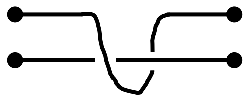
composed with

yields

$=$

Particle encircling another is a double braid. Particles at original positions, but worldlines remain linked.

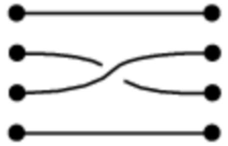
$\circ$ $=$



The composition of the braids σ and τ is written as στ.

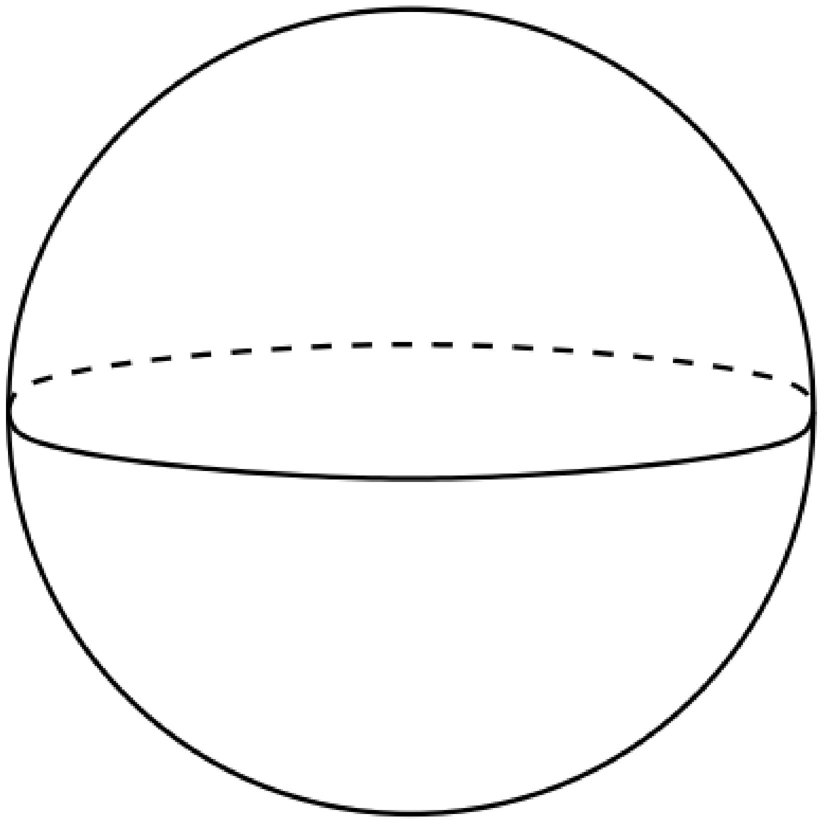
Aside While We're on Loops: Braiding, Exchange-Statistics, Dimensionality

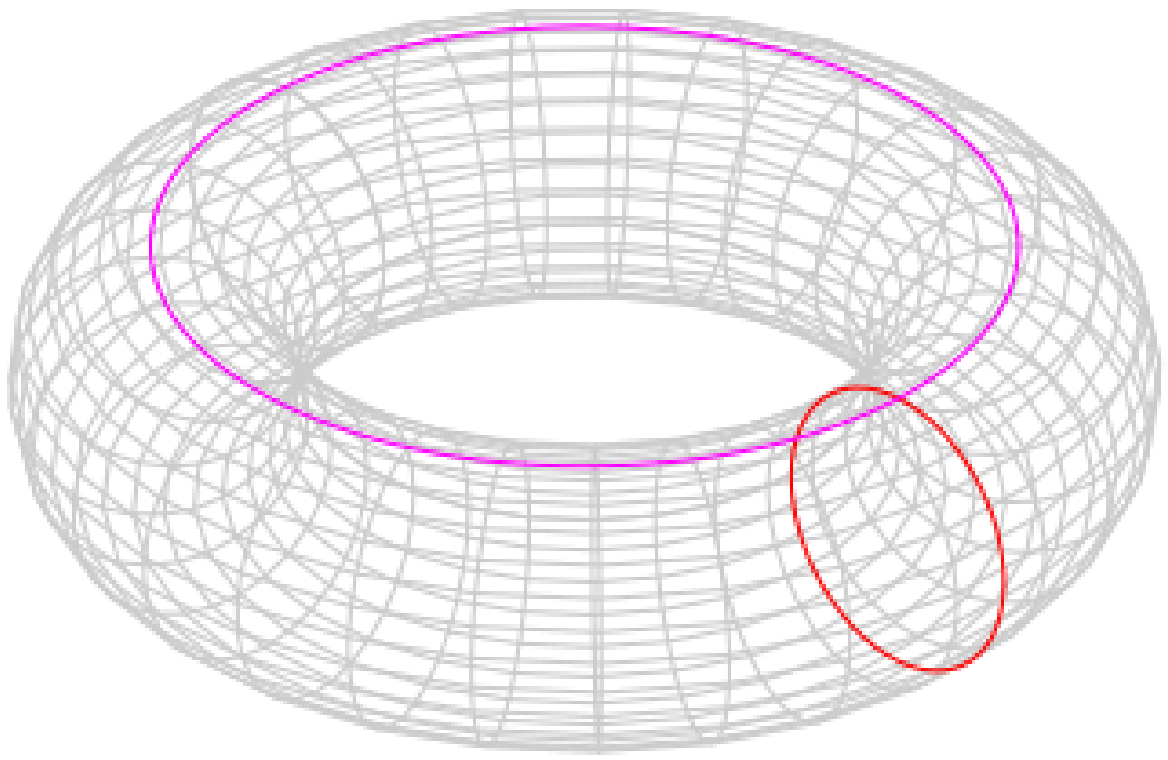
- Imagine a wavefunction of many identical particles of type a, b, c.
- $\Psi(a_1, \cdots, a_N, b_1, \cdots, b_M, c_1, \cdots, c_O) = \Psi\left(\vec{a}, \vec{b}, \vec{c}\right)$
  - $\mu_i = (\mu, x_i, y_i, z_i)$ means a particle of type μ at coordinates (x_i, y_i, z_i)
  - Since these are identical particles no observable, or measurable quantity, can differ if we ***exchange*** two particles of the same type
- Under exchange of *identical* particles
  - $|\Phi\rangle = \left|\Psi\left(a_2, a_1, \cdots, \vec{b}, \vec{c}\right)\right\rangle = \sigma_{a_2,a_1}\left|\Psi\left(\vec{a}, \vec{b}, \vec{c}\right)\right\rangle$
  - Equivalence under exchange implies:
    - $\left\langle\Psi\left(\vec{a}, \vec{b}, \vec{c}\right)\middle|\hat{O}\middle|\Psi\left(\vec{a}, \vec{b}, \vec{c}\right)\right\rangle = \left\langle\hat{O}\right\rangle_\Psi = \left\langle\hat{O}\right\rangle_\Phi$
    - $(+1)^2 = (-1)^2 = 1$ correspond to bosons and fermions

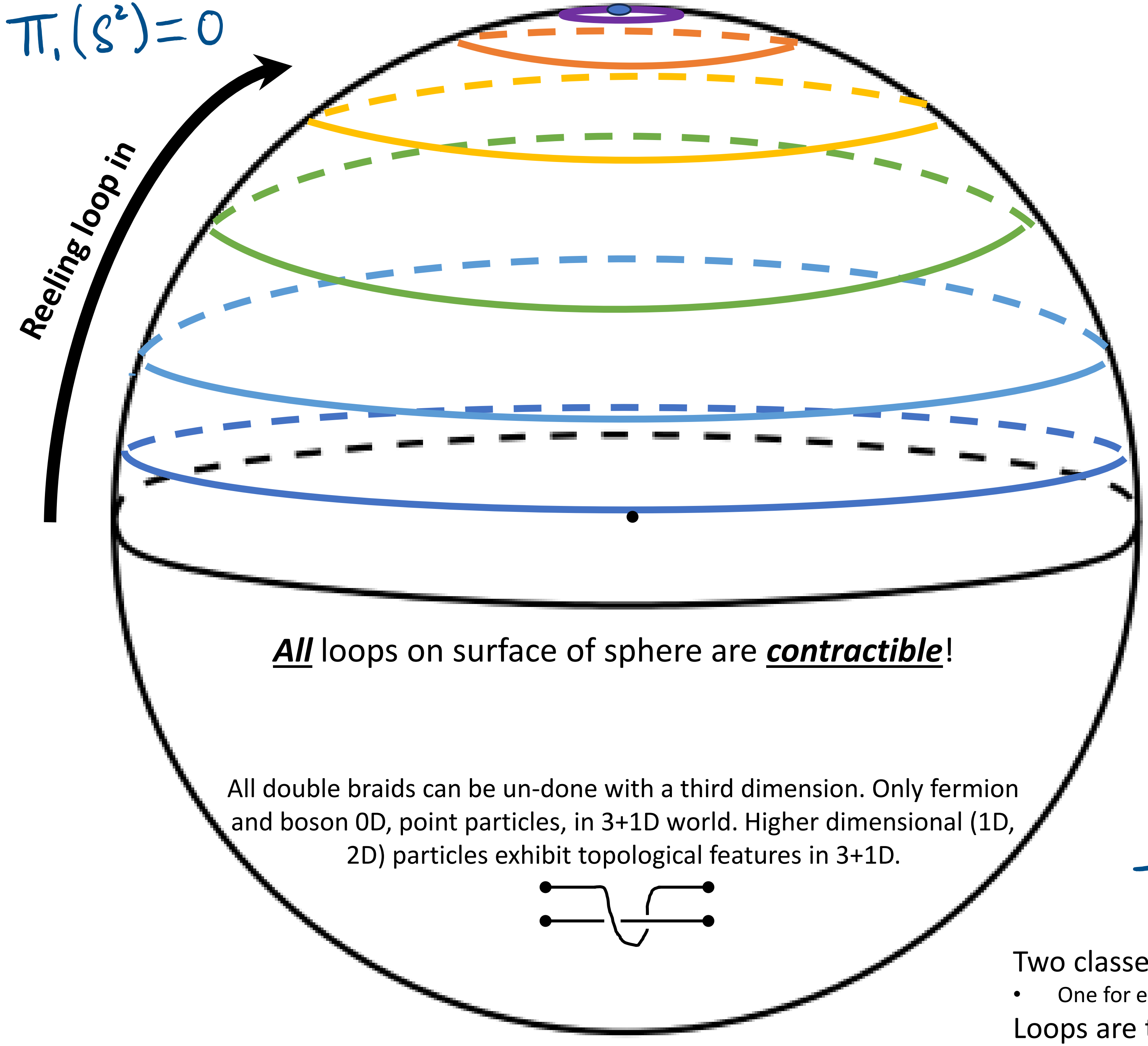


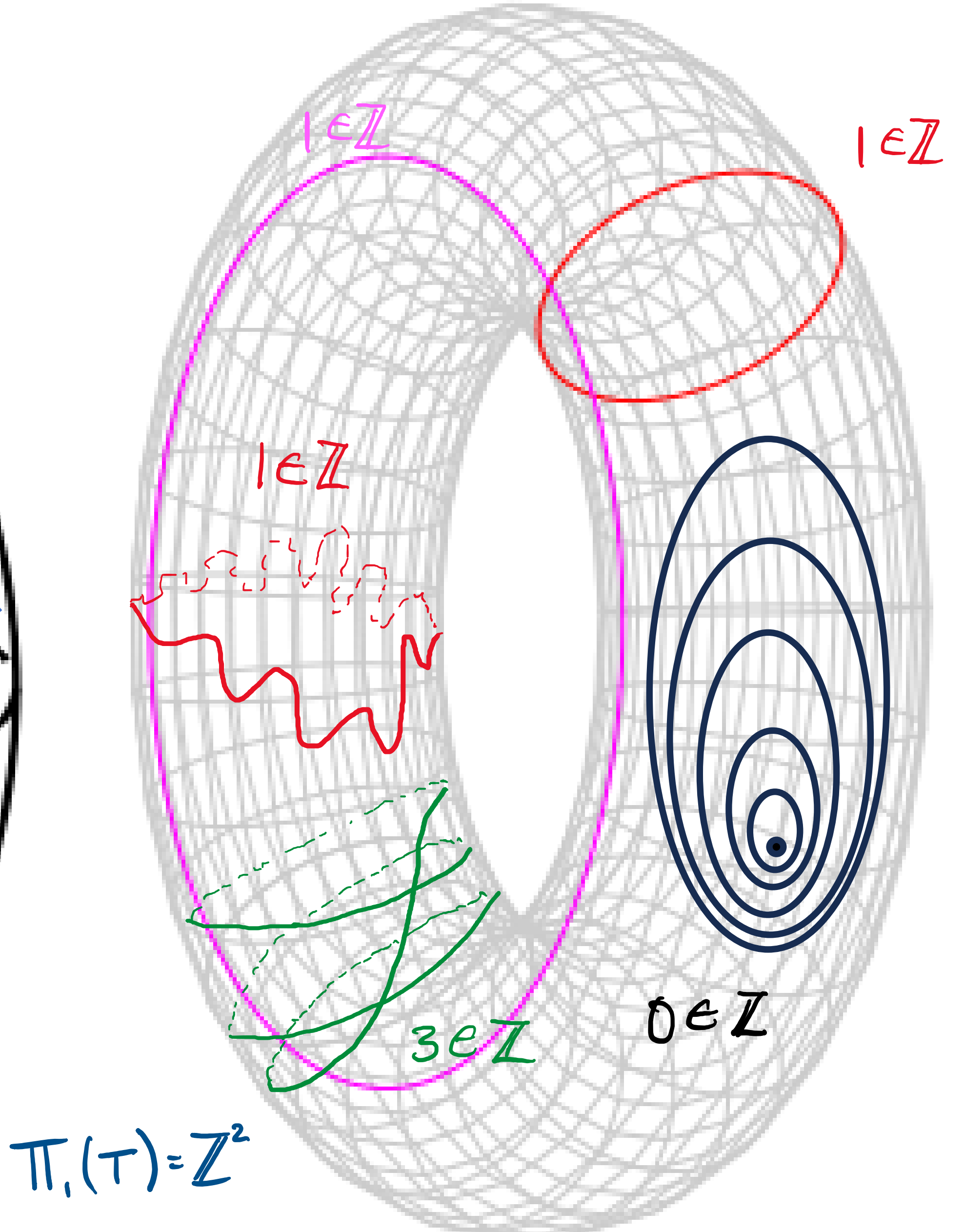
Homotopy and The Fundamental Group

- Basic idea is that topological spaces cover each other, and we can use the way they cover one another to differentiate between them.
- In the mathematical field of algebraic topology, the **fundamental group** of a topological space is the group of the equivalence classes under homotopy of the loops contained in the space $\pi_1(M)$









Two classes of **non**-***contractible***, topologically non-trivial, loops!

- One for each handle of the Torus

Loops are trivial if not encircling either handle of the torus

Classical Lattice Surgery: A Primer

Assignments:

1. Cut, twist, and glue/tape a sheet of paper to form a Möbius strip.
    1. Congrats on your first successful classical lattice surgery.
    2. Describe the topology
2. Cut the Möbius strip in half along the long, twisted handle.
    1. Describe the resulting topology.

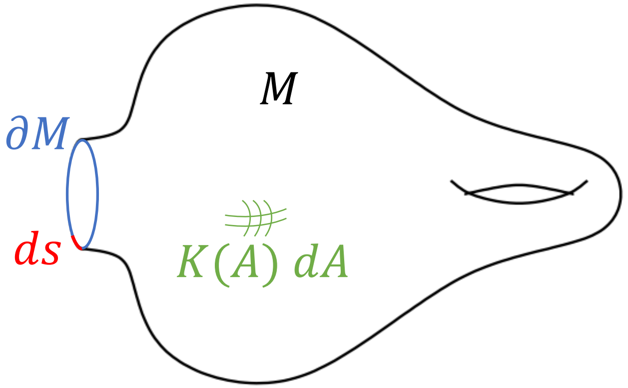
Homework:

3. Cut the Möbius strip, like in 2, but at the $\frac{1}{3}$ mark.

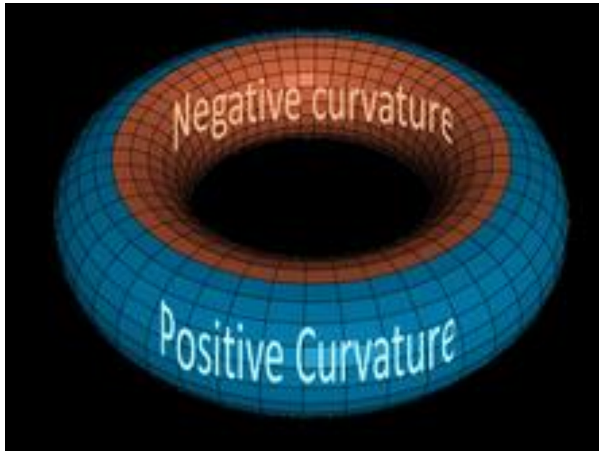
    Describe the resulting topology.

Topological Invariant #1: Euler Characteristic and the Gauss-Bonnet Theorem

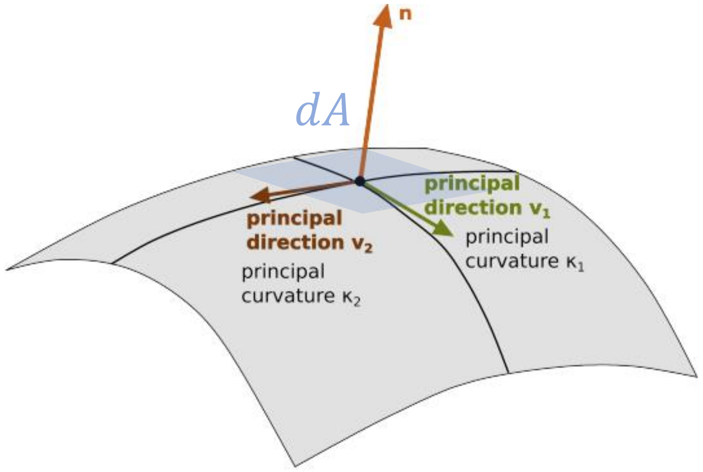
$$\int_{\partial M} k_g\, ds + \int_M K\, dA = 2\pi\chi(M)$$



Geodesic curvature k_g integrated over (little bits ds) of manifold M’s boundary ∂M



Gaussian curvature $K = \kappa_1\kappa_2$ integrated over of surface of manifold M



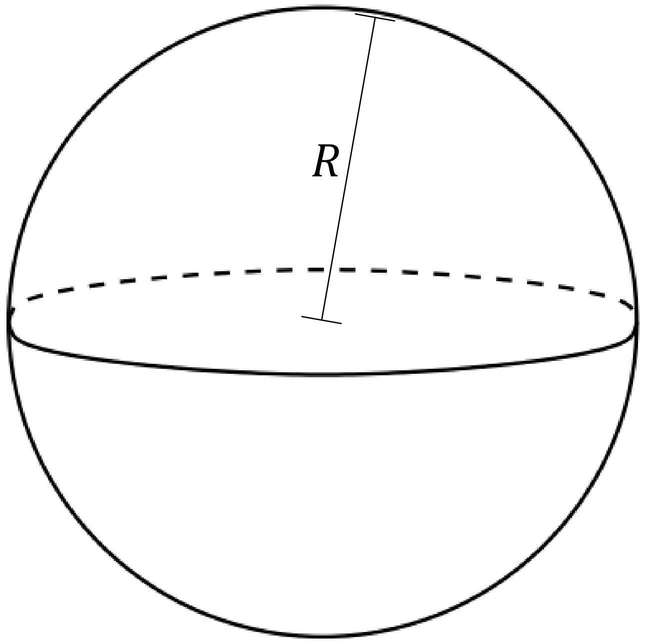
Euler characteristic $\chi(M)$ is a ***topological invariant*** which is invariant to bending and stretching of the manifold.

Any extra positive curvature somewhere is negated by negative curvature elsewhere.

$\chi(M)$ is a ***global*** property

$\chi(M) - 1$ is the ***genus***. I.e. the number of ***holes*** in a 2D surface embedded in 3D

Analytic Exercise # 1



Using $K(R,\theta,\phi) = \frac{1}{R} \cdot \frac{1}{R} = \frac{1}{R^2}$,
for a sphere of radius R, compute:

$$\int_0^{2\pi} R d\theta \int_0^{\pi} R sin\phi d\phi\, K = 2\pi\chi(M)$$

HW/test question(s):
What is the Euler characteristic of a sphere? What is its genus?

The topological theory of defects in ordered media*†

*Topologically trivial

N. D. Mermin

Laboratory of Atomic and Solid State Physics Cornell University, Ithaca, New York 14853

Aspects of the theory of homotopy groups are described in a mathematical style closer to that of condensed matter physics than that of topology. The aim is to make more readily accessible to physicists the recent applications of homotopy theory to the study of defects in ordered media. Although many physical examples are woven into the development of the subject, the focus is on mathematical pedagogy rather than on a systematic review of applications.

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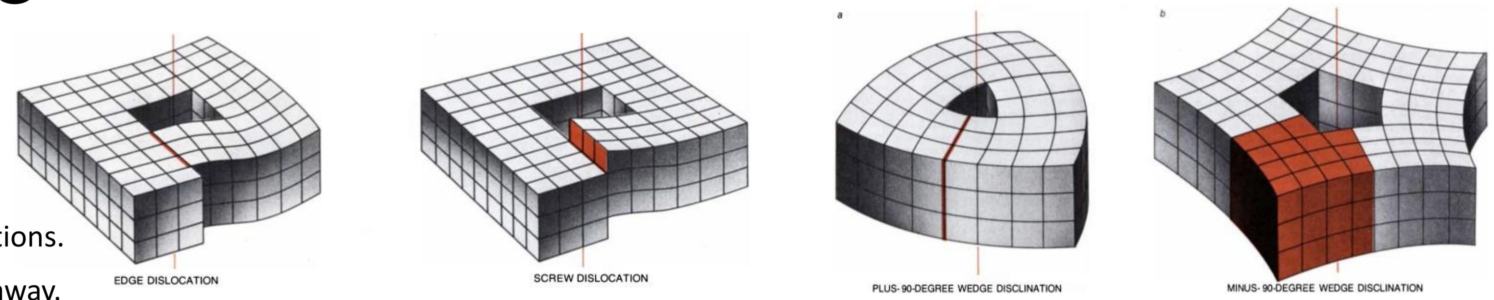
*A preliminary version of this paper appeared as Technical Report No. 3045 of the Materials Science Center of Cornell University. It has been substantially revised at the University of Sussex, with the partial financial support of the Science Research Council of Great Britain and with the generous help and hospitality of A. J. Leggett and D. F. Brewer. The revision has also been supported in part by the National Science Foundation under Grant No. DMR 77-18329 and through the Materials Science Center of Cornell University, Technical Report No. 4021.

†A complementary review of homotopy, from the point of view of a mathematician and field theorist, by Louis Michel is scheduled to appear in a future issue of Reviews of Modern Physics.

Topological *Defects*

Disclinations, William F. Harris: Scientific American 1977



$C_3 \leftarrow \quad C_4 \quad \rightarrow C_5$

Topological Defects

Defects which are stable and cannot destroyed purely local actions.

The defect may extend of to the end of the crystal or very far away.

e.g. Crystalline Defects

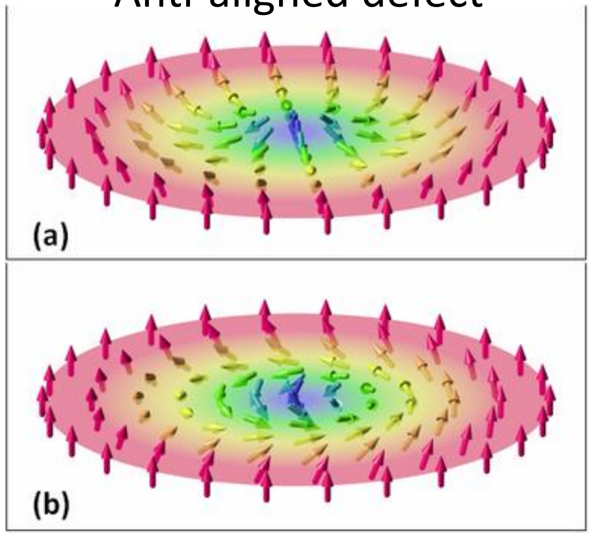
- Lateral dislocation
- Rotational disclination
- In 2D defect lines become *points*

And for, e.g. magnetic, vector fields

BKT Transition: vortex · anti-vortex = 0

We will find computational/information-theoretic analogies of this phenomenon tomorrow!

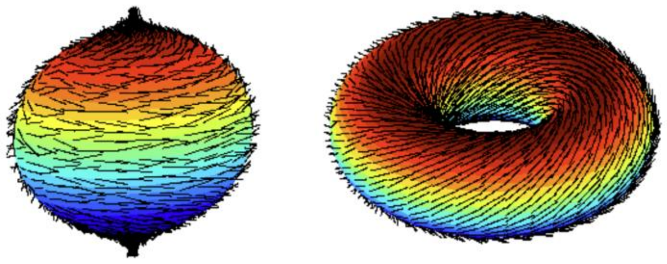
Anti-aligned defect



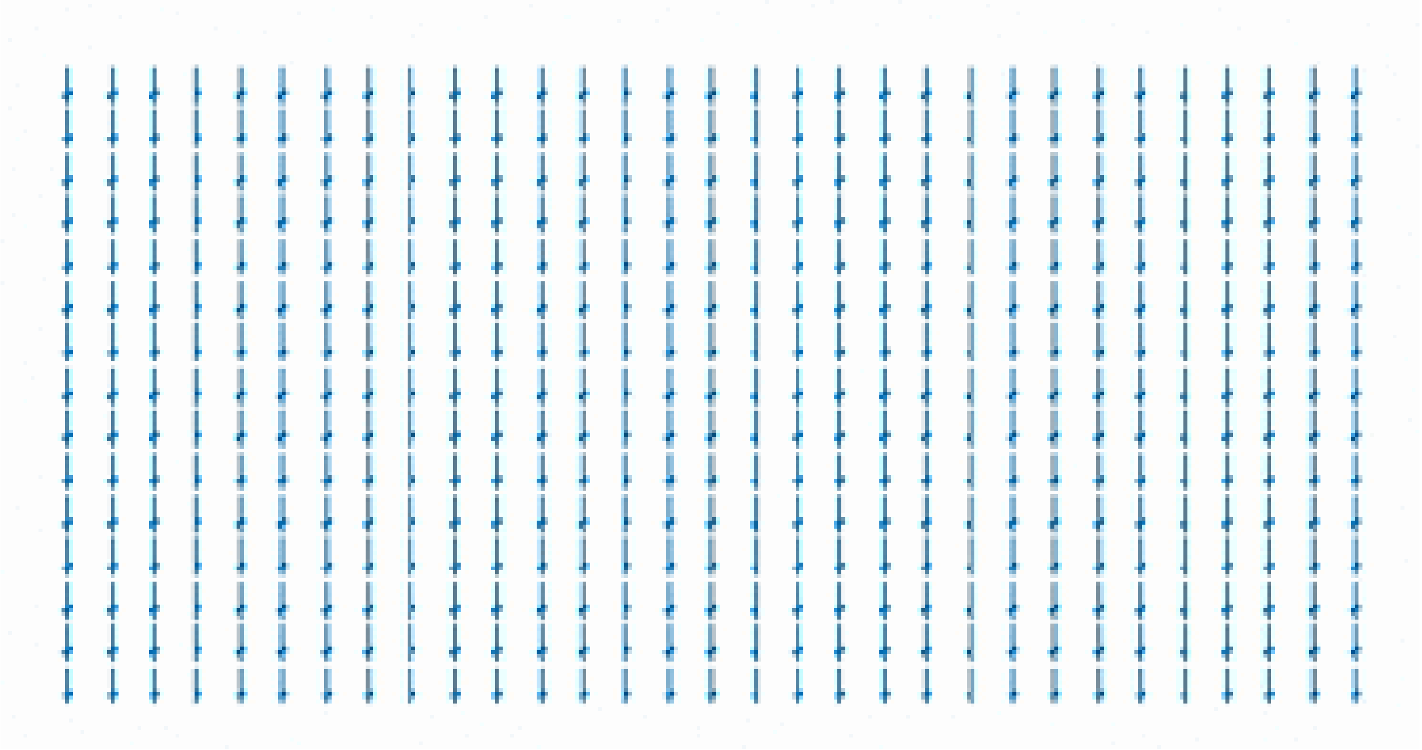
Twisted defect

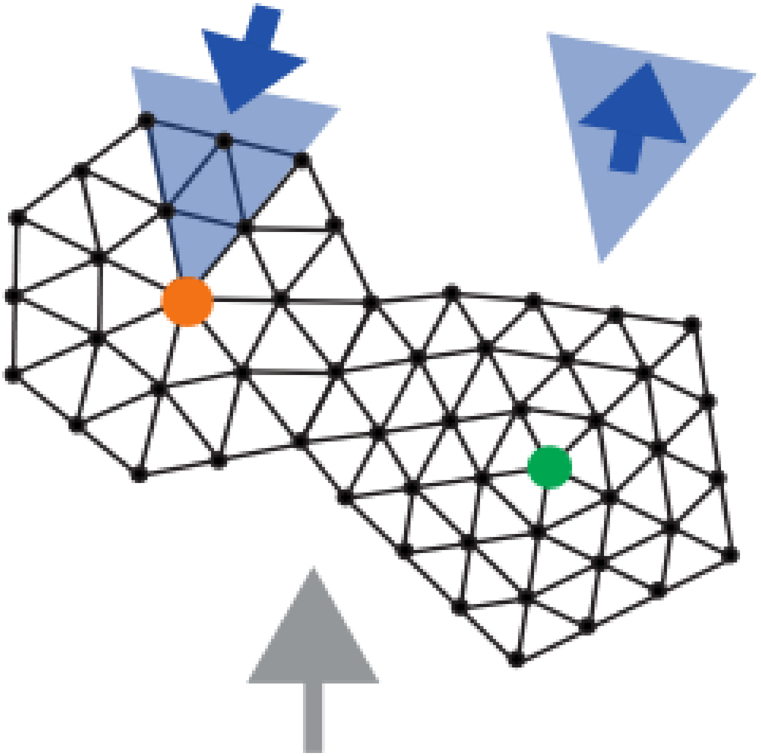
The hairy ball theorem

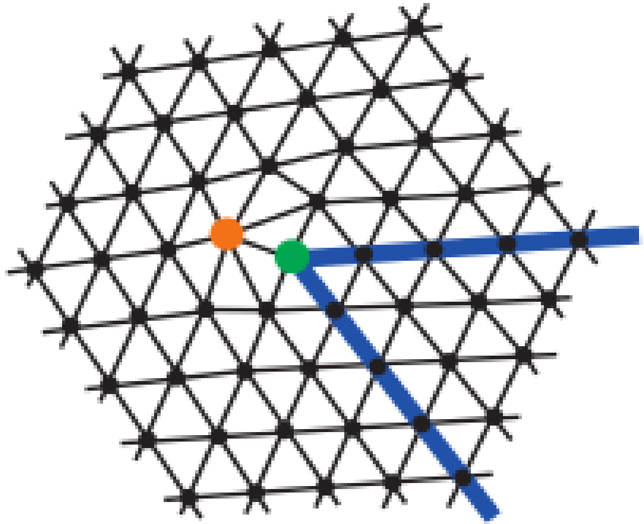
- "you can't comb a hairy ball flat without creating a cowlick"



Topological Defects
In topologically *trivial* systems







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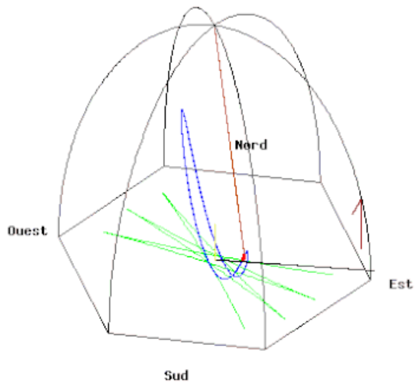
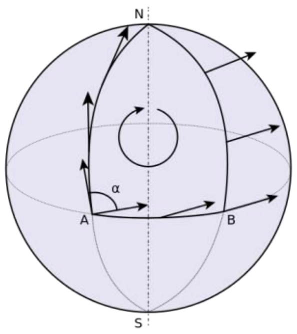
- https://www.ribbonfarm.com/2015/09/24/samuel-becketts-guide-to-particles-and-antiparticles/
- https://johncarlosbaez.wordpress.com/2016/10/07/kosterlitz-thouless-transition/



Not to scale

Parallel Transport and the Geometric Phase of Quantum Bands:



- Instantaneously acquiring a local phase from the rotation of the Earth's curvature ($S^2 \in \mathbb{R}^3$) and rotation.
  - Parallel transport is felt as a Coriolis force in our intertial co-rotating frame.
  - Angular speed due to rotation: $\omega = \frac{2\pi}{day} sin\phi$
- After each cycle the system returns to its original position with a velocity modified by the angular phase it acquired along the cycle's closed path.
- Skipping over many examples of topological waves/excitations/phenomena due to parallel transport.
  - **The unifying role of topology,** Mark Buchanan, https://www.nature.com/articles/s41567-020-1001-y

Topological Quantum Matter

- What is the phase acquired by a quantum state as it evolves (precesses) along some curve (energy potential)? Analogous expression for parallel transport is:
  - $A_\mu = i\langle\psi|\partial_\mu|\psi\rangle = i\langle\psi|\partial_\mu\psi\rangle$
    - This is the Berry **Connection**. It’s a complex number (amplitude) that tells us how the state (a vector) connects with its tangent vector. It’s a rate of change of the curve with respect to μ
  - $\gamma = \int_C d\boldsymbol{l} \cdot A_l$ is the Pancharantnam-Berry Phase
  - $\Omega_{\nu\mu} = \nabla_\nu \times A_\mu$
    - Additional contribution to electron velocity!
    - Given by the curvature of the quantum bands

The velocity operator in the q representation has the form $v(q,t)=\partial H(q,t)/\partial(\hbar q)$.[6] Hence, the average velocity in a state of given q is found to first order as

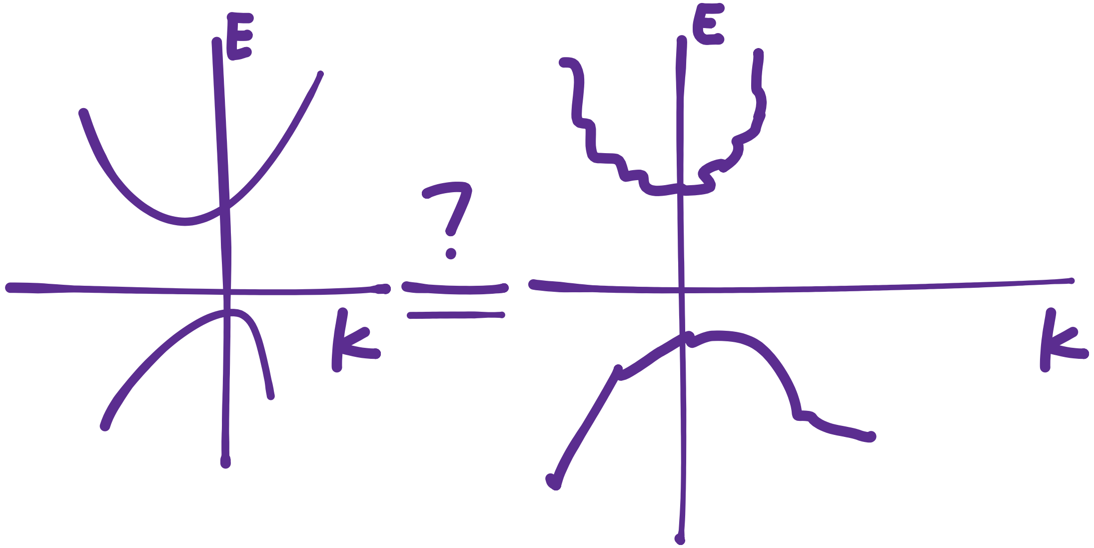
$$v_n(q) = \frac{\partial \varepsilon_n(q)}{\hbar \partial q} - i \sum_{n' \neq n} \left\{ \frac{\langle u_n|\partial H/\partial q|u_{n'}\rangle\langle u_{n'}|\partial u_n/\partial t\rangle}{\varepsilon_n - \varepsilon_{n'}} - \text{c.c.} \right\}, \tag{2.3}$$

where c.c. denotes the complex conjugate. Using the fact that $\langle u_n|\partial H/\partial q|u_{n'}\rangle = (\varepsilon_n - \varepsilon_{n'})\langle \partial u_n/\partial q|u_{n'}\rangle$ and the identity $\Sigma_{n'}|u_{n'}\rangle\langle u_{n'}|=1$, we find

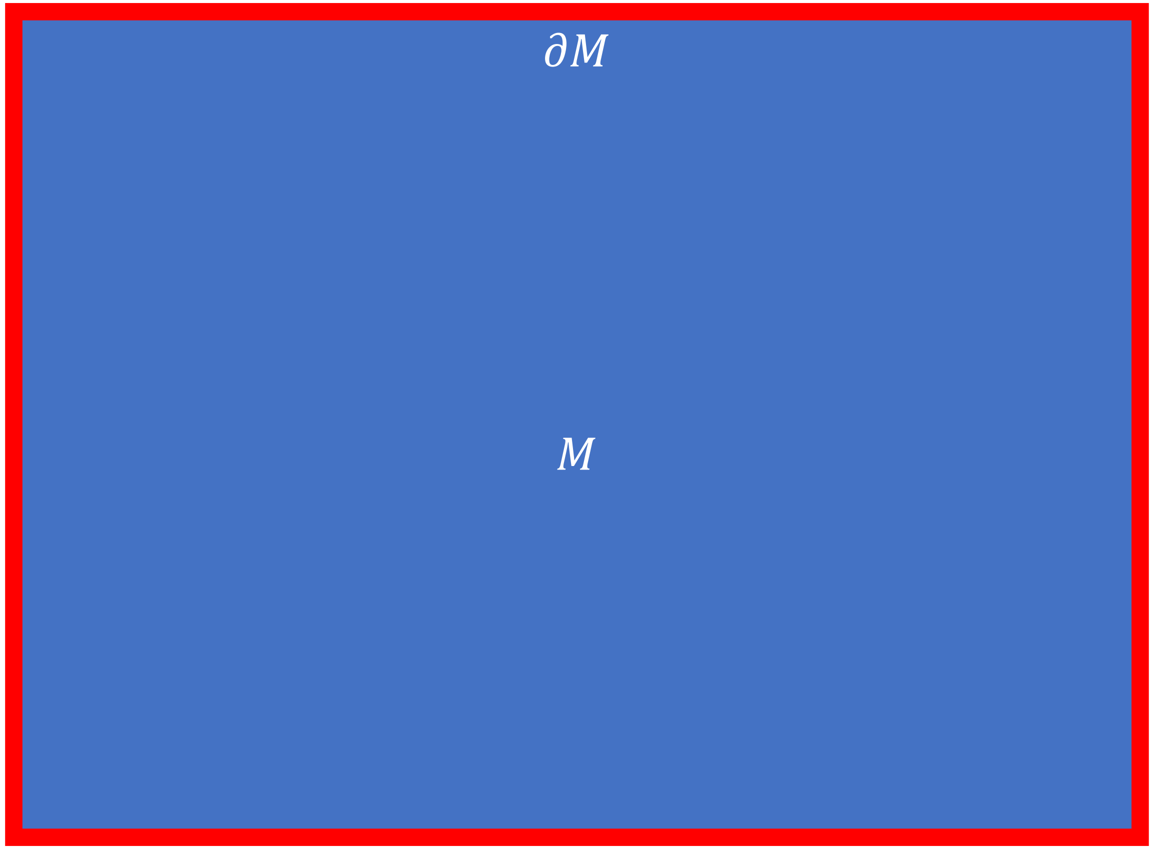
$$v_n(q) = \frac{\partial \varepsilon_n(q)}{\hbar \partial q} - i\left[\left\langle \frac{\partial u_n}{\partial q}\middle|\frac{\partial u_n}{\partial t}\right\rangle - \left\langle \frac{\partial u_n}{\partial t}\middle|\frac{\partial u_n}{\partial q}\right\rangle\right]. \tag{2.4}$$

Quantum Band Topology & the Holographic Bulk-Boundary Correspondence

Two energy bands are **topologically *equivalent*** if they can be deformed into one another *without* closing the (or opening a new) *gap*.



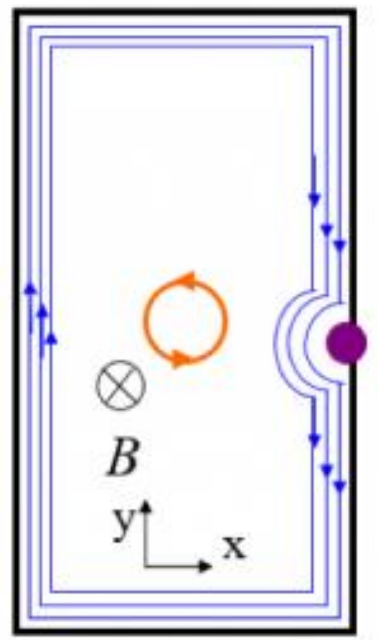
Boundary & Bulk



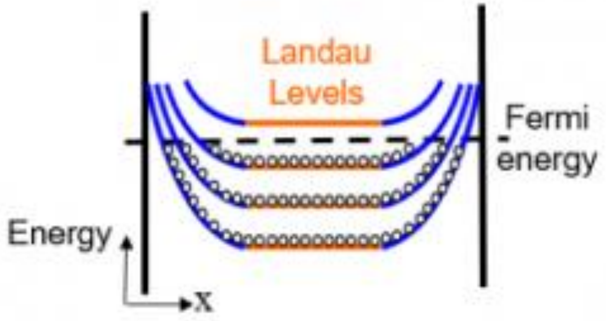
The topological bulk information is also encoded on the surface

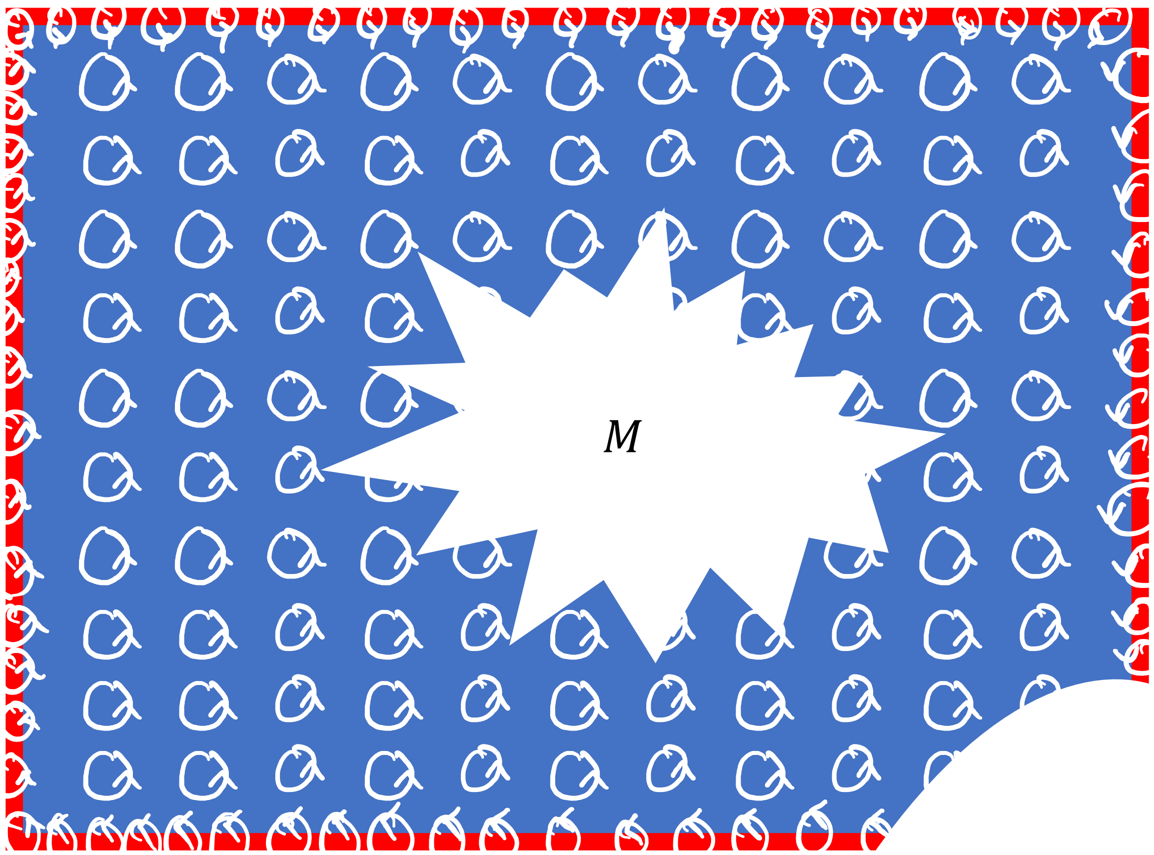
Altland & Zirnbauer, "Novel Symmetry Classes in Mesoscopic Normal-Superconducting Hybrid Structures" PhysRevB.55.1142

Information in the bulk is encoded on the boundary surface. A manifestation of Stokes theorem.



newshub.sustech.edu.cn/en/html/201904/12775.html





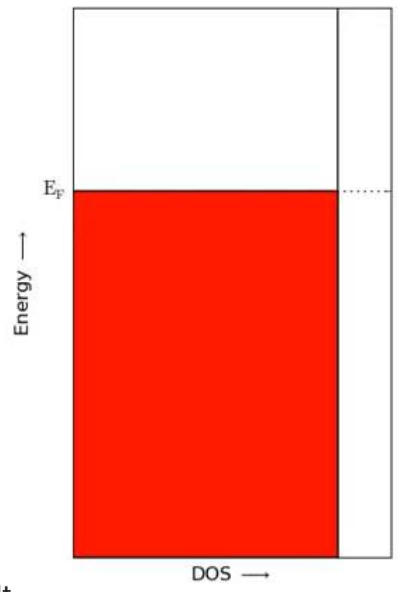
Bulk orbitals: localized! Electrons are confined, by the large magnetic field, to harmonically oscillate within a small region.

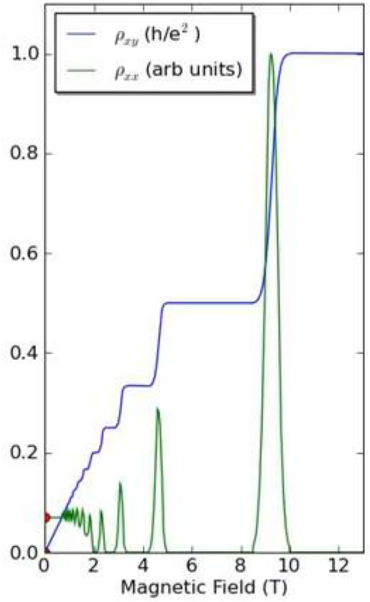
No particles collisions, but they would feel Coulomb repulsion (if we considered it)

Edge orbitals: cut-off by boundary. Forced to “skip” along edge. Whatever the edge may be!

Topological Quantum Edge States

- Bulk-boundary correspondence and edge states.
- Observe the following facts:
  - The number of quantized edge states is an integer
  - The number of edge states changes upon a topological phase transition
  - The same number (of edge states) can be found by integrating the Berry curvature over all occupied quantum states





The quantization of the Hall conductance ($G_{xy} = 1/R_{xy}$) has the important property of being exceedingly precise.[5] Actual measurements of the Hall conductance have been found to be integer or fractional multiples of $\frac{e^2}{h}$ to nearly one part in a billion. It has allowed for the definition of a new practical standard for electrical resistance, based on the resistance quantum given by the **von Klitzing constant** R_K. This is named after Klaus von Klitzing, the discoverer of exact quantization. The quantum Hall effect also provides an extremely precise independent determination of the fine-structure constant, a quantity of fundamental importance in quantum electrodynamics.

In 1990, a fixed conventional value $R_{K\text{-}90} = 25\,812.807\ \Omega$ was defined for use in resistance calibrations worldwide.[6] On 16 November 2018, the 26th meeting of the General Conference on Weights and Measures decided to fix exact values of h (the Planck constant) and e (the elementary charge),[7] superseding the 1990 value with an exact permanent value $R_K = \frac{h}{e^2} = 25\,812.807\,45\ldots\ \Omega$.[8]

Examples and In Class Exercises

- Python exercise: Edge Mode Exploration
- See attached python notebooks to compute:
  - Unpaired Majorana fermions in quantum wires
    - https://arxiv.org/abs/cond-mat/0010440
    - https://iopscience.iop.org/article/10.1070/1063-7869/44/10S/S29
  - SSH model edge modes

- Zoo of Topological Invariants
  - (0-dimensional) Qubit Exercise
    - Compute the SU(2) invariant as per Niu/Bernevig and earlier.

Conclusion

- Topology is ubiquitous
  - Also smooth, except when there are boundaries
  - Rich mathematical subject
  - Physically ubiquitous in classical phenomena
  - Also appear in topologically trivial quantum matter
- Quantum Topology is also ubiquitous
  - Curvature, winding, and covering invariants
  - Physical realizations
  - Bulk-boundary holographic duality
  - Robustness of excitations
- This is a foundation for what we will see tomorrow